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THE VALUATION OF COMPOUND OPTIONS*

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This paper presents a theory for pricing options on options, or compound options. The method can be generalized to value many corporate liabilities. The compound call option formula derived herein considers a call option on stock which is itself an option on the assets of the firm. This perspective incorporates leverage effects into option pricing and consequently the variance of the rate of return on the stock is not constant as Black–Scholes assumed, but is instead a function of the level of the stock price. The Black–Scholes formula is shown to be a special case of the compound option formula. This new model for puts and calls corrects some important biases of the Black–Scholes model.

1. Introduction

Almost any opportunity with a choice whose value depends on an underlying asset can be viewed as an option. A contract specifies the terms of the opportunity, or details what financial economists call the option's boundary conditions. Many opportunities have a sequential nature, where latter opportunities are available only if earlier opportunities are undertaken. Such is the nature of the compound option or option on an option.

Black and Scholes (1973) indicated in their seminal paper that most corporate liabilities may be viewed as options. After deriving a formula for the value of a call option, they discussed the pricing of a firm's common stock and bonds when the stock is viewed as an option on the value of the firm. In this setting, an option on the common stock is an option on an option. They also suggested that when a company has coupon bonds

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outstanding, the common stock and coupon bonds can be viewed as a compound option, and warrants and stocks that pay constant dividends can also be considered compound options. Geske (1977) has derived formulas for valuing coupon bonds and subordinated debt as compound options, while Roll (1977) has used this technique to value American options on stocks paying constant dividends. Recently, Myers (1977) has suggested that corporate investment opportunities may be represented as options. In that setting, common stock is again a compound option. Insurance policies with sequential premiums offer another application of the compound option technique. This paper develops the theory for dealing with compound option problems.

The main difficulty in using the Black–Scholes differential equation when dealing with compound options is that it assumes that the variance rate of the return on the stock is constant. However, with compound options this variance is not constant, but depends on the level of the stock price, or more fundamentally, on the value of the firm.

In section 2, the valuation equation for a call as a compound option is derived in continuous time, using a hedging argument. The solution contains an additional term than the Black–Scholes (1973) solution, which reflects the firm's debt position. It is this financial leverage which alters the total risk or volatility of the stockholder's equity as the market continuously revalues the firm's prospective cash flows. The derived formula has the desirable attributes of the Black–Scholes model in that it does not depend on knowledge of the expected return on either the stock or the firm's assets. It is shown that an alternate hedging approach, a risk neutral approach, and a discrete time approach all lead to the same result, and some similarities and differences of these approaches are offered. Comparative statics are also presented here. In section 3, a comparison to the Black–Scholes model shows it to be a special case of the compound option model. Changes in the equity value change the firm's leverage, and the stock's return variance is shown to be monotonic increasing with leverage in the compound option model. Thus, this model has the potential to correct several important biases of the Black–Scholes model. It is shown that the hedge ratio between the option and the stock is different for these two models, and that the Black–Scholes hedge is not riskless for any levered firm but instead leads to overinvestment in calls or under investment in stock. Section 4 considers relationships to other option pricing results. Here it is shown that the lognormal distribution for stock returns assumed by Black–Scholes/Merton is not consistent with the Modigliani–Miller theorem (M–M) and risky debt. However, in the compound option setting, the induced return distribution of stock prices, which cannot be lognormal, is consistent. Also, the compound option model's relation to the Cox–Ross (1975) constant elasticity of variance model and warrants as compound options are discussed.

2. The valuation equation

A formula for the value of a call option, C, as a compound option can be derived as a function of the value of the firm, V, if the firm's stock, S, can be viewed as an option on the value of the firm. The following setting describes this perspective. Consider a corporation that has common stock and bonds outstanding. Suppose the bonds are pure discount bonds, giving the holder the right to the face value, M, if the corporation can pay it, with a maturity of T years. Suppose the indenture of the bond stipulates that the firm cannot issue any new senior or equivalent rank claims on the firm, nor pay cash dividends or repurchase shares prior to the maturity of the bonds. Finally, suppose the firm plans to liquidate in T years, pay off the bonds, if possible, and pay any remaining value to the stockholders as a liquidating dividend.[1] Here the bondholders own the firm's assets and have given the stockholders the option to buy the assets back when the bonds mature. Now a call on the firm's stock is an option on an option or a compound option. This situation can be represented functionally as $C=f(S,t)=f(g(V,t),t)$, where t is current time. Therefore, changes in the value of the call can be expressed as a function of changes in the value of the firm and changes in time. If the value of the firm follows a continuous sample path, and if investors can continuously adjust their positions, a riskless hedge can be formed by choosing an appropriate mixture of the firm and call options on the firm's stock.

Merton (1973a) has shown that an American call option will not be exercised early if the underlying asset has no payouts. Thus, the stock depicted as an option on the value of the firm, will not be exercised early because the firm by assumption makes no dividend or coupon payments. Since Merton's proof does not rely on any distributional assumptions the compound call option on the stock will not be prematurely exercised either.

To derive the compound option formula for a call in continuous time, assume that security markets are perfect and competitive, unrestricted short sales of all assets with full use of proceeds is allowed, the risk-free rate of interest is known and constant over time, trading takes place continuously in time, and changes in the value of the firm follow a random walk in continuous time with a variance rate proportional to the square of the value of the firm. Thus, the return on the firm follows a diffusion described by the following stochastic differential equation formalized by Ito:

$$\mathrm{d}V/V=\alpha_V\,\mathrm{d}t+\sigma_V\,\mathrm{d}Z_V,$$

where α_V is the instantaneous expected rate of return on the firm per unit time, σ_V^2 is the instantaneous variance of the return on the firm per unit time,

[1] Most of these restrictions can be relaxed. In particular, the firm does not have to liquidate at date T, but could pay off the bonds and refinance.

and dZ_V is a mean zero normal random variable with variance dt, or a standard Gauss–Weiner process.

Since the call option is a function of the value of the firm and time, $C(V,t)$, its return also follows a diffusion process that can be described by a related stochastic differential equation,

$$dC/C = \alpha_C \, dt + \sigma_C \, dZ_C,$$

where α_C is the instantaneous expected rate of return per unit time on the call, σ_C^2 is the instantaneous variance of the return per unit time, and dZ_C is also a standard Gauss–Weiner process. Because of the functional relationship between C and V, α_C, σ_C, and dZ_C are explicitly related to α_V, σ_V, and dZ_V, and by employing either Ito's lemma or a Taylor's series expansion, the dynamics of the call option can be re-expressed as

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial V} dV + \frac{1}{2} \frac{\partial^2 C}{\partial V^2} V^2 \sigma_V^2 \, dt.$$

As Black–Scholes (1973) demonstrated, a riskless hedge can be created and maintained with two securities, in this case the firm and a call, which requires a net investment that earns the riskless rate of interest. Alternatively, Merton (1973a) showed that a three security riskless hedge portfolio containing an additional risk-free instrument can be created for zero net investment by using the proceeds from short sales and borrowings to finance the long position. Following this alternative, let n_1 be the instantaneous number of dollars invested in the firm, n_2 the instantaneous number of dollars invested in the call, and $n_3 \equiv -(n_1 + n_2)$ the instantaneous number of dollars invested in riskless debt. Now if dH is the instantaneous dollar return to the hedge portfolio, then

$$dH = n_1(dV/V) + n_2(dC/C) + n_3 r_F \, dt.$$

Substituting for the stochastic return on the firm and the call yields

$$dH = [n_1(\alpha_V - r_F) + n_2(\alpha_C - r_F)] \, dt + [n_1 \sigma_V + n_2 \sigma_C] \, dZ_V.$$

Since this portfolio requires zero net investment, if it could be made non-stochastic ($dZ_V = 0$), then to avoid arbitrage profits, the expected and realized return on this portfolio must be zero. Therefore, a strategy of choosing n_j^* so that $dZ_V = 0$ implies that $dH = 0$. A non-trivial solution ($n_j^* \neq 0$) exists if and only if $(\alpha_V - r_F)/\sigma_V = (\alpha_C - r_F)/\sigma_C$. Substituting for α_C and σ_C, and then simplifying yields the familiar partial differential equation,

$$\frac{\partial C}{\partial t} = r_F C - r_F V \frac{\partial C}{\partial V} - \frac{1}{2} \sigma_V^2 V^2 \frac{\partial^2 C}{\partial V^2}. \tag{1}$$

Eq. (1) for a call option on the firm's stock as a function of V and t is subject to a boundary condition at $t=t^*$, the expiration date of this option. The value of the call at expiration is either zero if the stock price, $\tilde{S}_{t^*}$, is less than or equal to the exercise price, K, or is equal to the difference between the stock price and the exercise price if the stock price is greater than the exercise price. Algebraically $C_{t^*}=\max(0, S_{t^*}-K)$. From the perspective of the stock as an option of the value of the firm, this boundary poses a problem not previously encountered in option pricing. The stochastic variable determining the option's value in (1) is not the stock price, $\tilde{S}$, which enters the boundary condition, but instead is the value of the firm. However, since the stock is an option on the value of the firm, it follows a related diffusion and by again using either Ito's lemma or a Taylor's series expansion its dynamics can be expressed as a function of $\tilde{V}$ and t as

$$\mathrm{d}S=\frac{\partial S}{\partial t}\,\mathrm{d}t+\frac{\partial S}{\partial V}\,\mathrm{d}V+\frac{1}{2}\frac{\partial^2 S}{\partial V^2}\,V^2\sigma_V^2\,\mathrm{d}t.$$

By constructing a similar hedge between the stock, the firm, and a riskless security, the stock's equilibrium path can be described by the following similar partial differential equation:

$$\frac{\partial S}{\partial t}=r_F S-r_F V\frac{\partial S}{\partial V}-\frac{1}{2}\sigma_V^2 V^2\frac{\partial^2 S}{\partial V^2}. \tag{2}$$

The boundary condition for eq. (2) at date T, the date the firm's pure discount bonds mature, is that $S_T=0$ if the value of the firm is less than or equal to the face value of the debt, or if the value of the firm is greater than the face value of the debt, S_T is equal to the difference between the value of the firm and the face value of the debt. The solution to eq. (2) subject to this boundary condition that $S_T=\max(0, V_T-M)$ is independent of eq. (1) and is the well known Black–Scholes equation

$$S=VN_1(k+\sigma_V\sqrt{T-t})-M\,\mathrm{e}^{-r_F(T-t)}N_1(k), \tag{3}$$

where

$$k=\frac{\ln(V/M)+(r_F-1/2\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}},$$

S = current market value of the stock,
V = current market value of the firm,
M = face value of the debt,
r_F = the risk-free rate of interest,
σ_V^2 = the instantaneous variance of the return on the assets of the firm,
t = current time,
T = maturity date of the debt,
$N_1(\cdot)$ = univariate cumulative normal distribution function.

Eq. (1) is more difficult to solve than (2) because its boundary condition depends on the solution to eq. (2). Essentially both partial differential eqs. (1) and (2) describing the equilibrium paths of the call option and the stock, subject to their respective boundary conditions, must hold simultaneously in the solution to eq. (1). The exercise decision at expiration of the call option on the stock, which depends on the relationship between the stock and the exercise price, can be characterized by a relationship between the value of the firm and the exercise price. Thus, at date $t=t^*$, the value of the firm that makes the holder of an option on the stock indifferent between exercising and not exercising the option is the solution to the integral equation $S_{t^*}-K=0$, where $\hat{S}_{t^*}$ is given in eq. (3) and $\tau=T-t^*$. This partitions the probability measure over the value of the firm at $\bar{V}$, which is defined as that value of the firm which solves the integral equation $S_{t^*}-K=0$. For values of the firm less than $\bar{V}$ the call option on the stock will remain unexercised, while if the value of the firm is greater than $\bar{V}$, the option will be exercised. Given these two partial differential eqs. (1) and (2), and their boundary conditions, the following solution for the value of the compound call option can be found either by Fourier transforms or by separation of variables.[2]

Theorem. *Assume that investors are unsatiated, that security markets are perfect and competitive, that unrestricted short sales with full use of proceeds is allowed, that the risk-free rate of interest is known and constant over time, that trading takes place continuously in time, that the firm has no payouts, that changes in the value of the firm follow a random walk in continuous time with a variance rate proportional to the square root of the value of the firm, and that investors agree on this variance* σ_V^2, *then*

$$C=VN_2(h+\sigma_V\sqrt{\tau_1},k+\sigma_V\sqrt{\tau_2};\sqrt{\tau_1/\tau_2})-M\,e^{-r_F\tau_2}N_2(h,k;\sqrt{\tau_1/\tau_2})-K\,e^{-r_F\tau_1}N_1(h), \tag{4}$$

where

$$h=\frac{\ln(V/\bar{V})+(r_F-\frac{1}{2}\sigma_V^2)\tau_1}{\sigma_V\sqrt{\tau_1}},$$

$$k=\frac{\ln(V/M)+(r_F-\frac{1}{2}\sigma_V^2)\tau_2}{\sigma_V\sqrt{\tau_2}},$$

$\bar{V}$ = *that value of* V *such that*

$$S_\tau-K=VN_1(k+\sigma_V\sqrt{\tau})-M\,e^{-r_F\tau}N_1(k)-K=0 \quad \textit{where} \quad \tau=T-t^*,$$

[2]This problem can also be solved by the Cox–Ross (1975) technique of assuming a risk-neutral set of preferences, or by Rubinstein's (1976) approach to discounting uncertain income streams. For a third general approach to these valuation problems see Garman (1976).

and the notation not previously specified is: C = current value of the call option, and $N_2(\cdot)$ = bivariate cumulative normal distribution function with h and k as upper integral limits and $\sqrt{\tau_1/\tau_2}$ as the correlation coefficient, where $\tau_1 = t^* - t$ and $\tau_2 = T - t$.

Proof. See appendix.

There are several alternate ways to derive the formula for the compound call option. All approaches are conceptually valuation by duplication. The essence of duplication as noted by Ross (1978) is that any asset of unknown value can be comparatively priced by finding or creating an 'identical' asset whose value is either known or can be determined.[3] An option can be readily valued by duplication because it is a derivative asset, or an asset whose value is derived from the optioned asset. The cash flows of the option can be duplicated by a hedge containing the optioned asset and a riskless security. An alternative to the previous hedge between the call and the firm for deriving the compound option formula would be the more traditional hedge between the call and the stock. If the stock follows a diffusion process, then the maintenance of a riskless hedge leads to the familiar Black–Scholes (1973) partial differential equation

$$\frac{\partial C}{\partial t} = r_F C - r_F S \frac{\partial C}{\partial S} - \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 C}{\partial S^2}. \tag{5}$$

However, if the stock is considered to be an option on the value of the firm, then with the previous assumptions eq. (3) relating S and V results. This equation implies that both S and σ_S are functions of V and t, and that $\sigma_S(V,t) = \{(\partial S/\partial V)(V/S)\}\sigma_V$. Thus, σ_S is not constant as Black–Scholes/Merton assumed,[4] but instead is a particular function of $S(V,t)$. Solving eq. (5) subject to the boundary condition for the option at $t = t^*$ and using the functional relationship between $S(V,t)$ and $\sigma_S(S(V),t)$ also yields eq. (4).[5]

Later Cox–Ross (1975) recognized that if a riskless hedge could be created and maintained then the transformation solution to the partial differential equation was not necessary. They argued that since no explicit use was made of preferences by Black–Scholes/Merton any set of preferences consistent with the distributional assumptions would be satisfactory. In particular, in a risk-neutral world where all assets earn the same expected rate of return, the riskless rate, the current value of an option is the following riskless discounted expected value of the option at expiration:

[3]An obvious dilemma occurs when valuing the most basic set of securities.

[4]See section 3 of this paper for elaboration about $\sigma_S(V,t)$.

[5]This result implies that the Black–Scholes equation with the 'proper' variance and stock price yields the compound option equation, a fact which enhances the significance of implicit volatility estimates.

$$C = e^{-r_F(t^*-t)} E\{\max(S_{t^*} - K, 0)\}. \tag{6}$$

Given the assumed relation between the stock and the firm, if the conditional distribution for the value of the firm at the option's expiration date is known, $F(V_{t^*}|V)$, then substituting from eq. (3) for $S(V,t^*)$, eq. (6) becomes

$$C = e^{-r_F\tau_1}\left\{\int_{\bar{K}}^{\infty} VN_1(k+\sigma_V\sqrt{\tau_1-\tau_2})F(V_{t^*}|V)\,dV - \int_{\bar{K}}^{\infty} M\,e^{-r_F(\tau_2-\tau_1)}N_1(k)F(V_{t^*}|V)\,dV - \int_{\bar{K}}^{\infty} KF(V_{t^*}|V)\,dV\right\}. \tag{7}$$

Evaluating these integrals yields the compound option eq. (4).

Recently Rubinstein (1976) developed a discrete time, preference specific, general equilibrium approach to valuing uncertain income streams and then used this to show that the Black–Scholes/Merton option pricing equation does not depend on the maintenance of a riskless hedge. He demonstrates that under certain conditions individual's demands can be aggregated so that the market's return space is spanned by an identifiable linear operator or vector of contingent claims which allows valuation by duplication. Letting $P(s(t))$ be a set of continuous random variables, not necessarily unique but containing the same price and probability information for all securities, Rubinstein shows that the value of a European call option on stock with the previous boundary condition is equal to the conditional expectation

$$C = E[(S_{t^*} - K)P(s(t))|S_{t^*} > K]. \tag{8}$$

When stock is considered an option on the firm, its value is the conditional expectation $S = E[V_T - M)P(s(t))|\tilde{V}_T > M]$. With the major assumptions on tastes and beliefs of Constant Proportional Risk-Averse utility functions (CPRA) and joint lognormality between $\tilde{P}$ and $\tilde{V}$, and with proper stationarity conditions, eq. (3) for S obtains.[6] Substituting this result into eq. (8) yields the following integral equation:

$$C = \int_{\bar{K}}^{\infty} VN_1(k+\sigma_V\sqrt{\tau_2-\tau_1})PF(P,V)\,dP\,dV$$

[6]See Rubinstein (1976) for further details. Although individuals are not required to have identical beliefs, a representative individual must exist. Here $P(s(t))$ is shortened to P. The discrete version of eq. (3) is similar except $e^{-r_F t}$ is replaced by r_F^{-t}. The joint lognormality assumption is not uncommon in finance. See Merton (1973b) for another example of this assumption.

$$-\int_{K}^{\infty} M r_F^{-(\tau_2-\tau_1)} N_1(k) PF(P,V)\,\mathrm{d}P\,\mathrm{d}V - \int_{K}^{\infty} KPF(P,V)\,\mathrm{d}P\,\mathrm{d}V, \tag{9}$$

where $F(P, V)$ is the joint probability distribution function of $\tilde{P}$ and $\tilde{V}$. Evaluating these integrals yields the discrete version of the compound option eq. (4).

One important distinction between the discrete time and continuous time derivations of option valuation formulas pertains to investor's expectations of the variance. Agreement about σ^2 is necessary in the continuous time models but not in the discrete time version if heterogeneous investor's beliefs can be meaningfully aggregated. The continuous time model collapses if there is the slightest disagreement about the variance. The ability to create a riskless hedge will induce those in disagreement to take infinite positions, and no market price equilibrium can be achieved. It is true that if by assumption all investors use the same past data the same way to estimate the variance there will be no disagreement. However, in the discrete formulation, as long as a representative individual exists, instead of using past price changes, investors may have possibly different priors about the variance which on 'average' are exactly correct, and the discrete model will be correct. This is because the discrete time approach does not depend on the formation and maintenance of a riskless hedge.[7]

The response of the compound option model to changes in the value of its arguments conforms to some but not all of the restrictions placed on the option price by the arguments of Merton:[8]

(1) As the value of the firm rises so does the call value

$$\partial C/\partial V = N_2(h+\sigma_V\sqrt{\tau_1},\ k+\sigma_V\sqrt{\tau_2};\ \sqrt{\tau_1/\tau_2}) \equiv N_2(\cdot) > 0. \tag{10}$$

Although increases in the value of the firm are divided, but not usually proportionally, between the debt and the equity, any increase in V increases the expected payoff to the option.

(2) As the face value of the debt increases, the call value falls,

$$\partial C/\partial M = -\mathrm{e}^{-r_F\tau_2} N_2(h;k;\sqrt{\tau_1/\tau_2}) < 0. \tag{11}$$

Even though the increased leverage raises the variance of the stock, σ_s^2, which increases the call price, the reduced equity value lowers the call price, and this first order equity effect dominates the second order variance effect.

[7]If individuals are Bayesian and realize there is estimation risk in $\tilde{\sigma}$ then they would not take infinite positions, but this is not consistent with the continuous time assumptions.

[8]See Merton (1973a, pp. 143–150). Use Liebnitz's rule for differentiation of an integral equation here. The intuition of these partial derivatives works through the stock price, S, which is not an explicit argument of eq. (4).

(3) As the time to maturity of the debt increases, the call price increases,

$$\frac{\partial C}{\partial T}=\frac{N_2(\cdot)}{N_1(k+\sigma_V\sqrt{\tau_2})}M\,\mathrm{e}^{-r_F\tau_2}\left[N_1'(k)\frac{\sigma_V}{2\sqrt{\tau_2}}+r_F N_1(k)\right]>0. \tag{12}$$

This reduces the present value of the debt, reducing the leverage and again the increased equity value dominates the reduced equity risk.[9]

(4) As the riskless rate of interest rises, the call price rises,

$$\frac{\partial C}{\partial r_F}=\frac{N_2(\cdot)}{N_1(k+\sigma_V\sqrt{\tau_2})}M\tau_2\,\mathrm{e}^{-r_F\tau_2}N_1(k)>0. \tag{13}$$

When the riskless interest rate rises the present value of the debt and of the option's exercise price fall, both increasing C. Even though σ_s falls as S rises, which should decrease C, the first order equity effect again dominates.

(5) As the variance rate of the firm rises, so does the call price,

$$\frac{\partial C}{\partial \sigma_V^2}=\frac{N_2(\cdot)}{N_1(k+\sigma_V\sqrt{\tau_2})}M\,\mathrm{e}^{-r_F\tau_2}N_1'(k)\frac{\sqrt{\tau_2}}{2\sigma_V}>0. \tag{14}$$

The increased variance rate of the firm raises the value of the equity as an option on the firm, which increases the cell value.

(6) As the call's exercise price rises, the call price falls,

$$\partial C/\partial K=-\mathrm{e}^{-r_F\tau_1}N_1(h)<0. \tag{15}$$

By dominance the value of the call in every state after K increases is less than or equal to its value before the exercise price change.

(7) As the time to expiration increases, the value of the call rises,

$$\partial C/\partial t^*>0. \tag{16}$$

This reflects the decreased present value of the future exercise price as t^* increases.

3. Comparison to the Black–Scholes model

The Black–Scholes model is a special case of the compound option model. To see this note that eq. (4) for valuing a call option as a compound option reduces to the Black–Scholes equation whenever the call is written on the

[9] $N'(k)\equiv(1/\sqrt{2\pi})\mathrm{e}^{-k^2/2}=$ the standard normal density at k.

equity of an unlevered firm. This occurs in eq. (4) whenever the present value of the firm's debt is zero, or when $T=\infty$ or $M=0$. In either case the stockholder's option to repurchase the firm from the bondholders disappears. Furthermore, if the option on the stock expires coincident with the maturity of the debt ($t^*=T$), the second option merges with the first, and the exercise price of this then simple option is the sum of the face value of the debt and the striking price of the option ($M+K$).

This result for pricing compound options incorporates the effects of short or long term changes in the firm's capital structure on the value of a call option. To see this notice that as the stock price changes, if the firm does not react, the debt-equity ratio of the firm changes, which should affect the riskiness of the firm's stock.[10] Whereas the Black–Scholes model assumes that the variance of the stock's return is not a function of the stock price, in the compound option model the variance of the return on the stock is inversely related to the stock price. As the stock price falls (rises), the firm's debt-equity ratio rises (falls), and this increased (decreased) risk is reflected by a rise (fall) in the variance of the returns on the stock. This can be demonstrated explicitly by taking the instantaneous covariance of the instantaneous return on the stock with itself and noting that changes in the stock price are perfectly correlated with changes in the value of the firm. So the instantaneous standard deviation of the return on the stock is, as given previously,

$$\sigma_S=\left(\frac{\partial S}{\partial V}\frac{V}{S}\right)\sigma_V=\varepsilon_S\sigma_V, \tag{17}$$

where $\varepsilon_S\equiv((\partial S/\partial V)(V/S))$ is the elasticity of the stock price with respect to the value of the firm. The partial derivative of the instantaneous standard deviation of the stock's return with respect to the stock price is

$$\frac{\partial\sigma_S}{\partial S}=-\frac{V}{S^2}\left(\frac{\partial S}{\partial V}\right)\sigma_V=-\frac{V}{S^2}N_1(k+\sigma_V\sqrt{\tau_2})\sigma_V<0.$$

Thus, in the short run, when fluctuations in the stock price are the main determinants of variations in the debt-equity ratio, percentage changes in the stock's return will be larger when prices have fallen than when they have risen. Since the value of an option is monotonic increasing in the volatility of the optioned asset, if the stock price has fallen (risen), the increased (decreased) variance of the returns on the stock will act to raise (lower) the price

[10]For expositional purposes of this argument, assume throughout that the firm does not react to stock price changes. Also M does not equal zero even if the firm has no long term debt because the liability side of most firm's balance sheets includes short term debt.

of the option on the stock. Thus, variations in the firm's capital structure induced by changes in the value of the firm as the market continuously revalues the firm's prospective cash flows are transmitted through the variance of the stock to affect the price of an option on the stock.

Since the variance of the stock is a function of the stock price in the compound option model, it is accordingly a function of all of the variables which determine the price of the stock, including T, the maturity date of the firm's debt. As the time to expiration of the option on the stock decreases, the life of the stock as an option on the value of the firm is also decreasing. Because the price of any call option is monotonic increasing in time to expiration, as T decreases, S does also.[11] This decrease of S causes an increase in the firm's debt-equity ratio, increasing the riskiness of the return on the firm's stock. The increase in the variance of the stock will act to increase the value of an option on the stock.

These leverage effects introduced into option pricing by the compound option model are not without empirical costs. One advantage of the Black–Scholes option pricing model is that only five input variables are required to predict option prices: $C^{BS}=f(S,\sigma_s,r_F,t^*,K)$. All of these variables are either known or directly observable except σ_s, the instantaneous variance of the stock return. The compound option formula requires seven input variables: $C^{CO}=g(V,\sigma_V,r_F,t^*,T,K,M)$. The two extra variables necessary to capture the leverage effects are M, the face value of the debt, and T, the maturity date of the debt. Three of these variables, r_F, t^*, and K, are directly observable and the other four can be computed. V and σ_V can be found either by defining $V=S+B$, where B is the market value of the firm's debt, and using empirical data, or by solving for V and σ_V from past stock price data, using $S(V,\sigma_V)$ and $\sigma_s(V,\sigma_V)$.[12] The face value of the firm's debt, M, and the maturity of the debt, T, can be read directly from the balance sheet, or for firms with more complex capital structures, surrogates can be constructed.[13] As in the Black–Scholes model, the significant unobservable variables not necessary for pricing compound options are the expected rates of return on the firm, the stock, and the option, and any measure of market risk aversion.

[11]The

$$\partial S/\partial T=M\,e^{-r_F\tau_2}\{(\sigma_V/2\sqrt{\tau_2})N_1'(k)+N_1(k)r_F\}>0.$$

Unlike all American options, European puts are not monotonic increasing in time to expiration.

[12]It may be better to estimate r_F since it does change stochastically. In addition, estimating the variance implicitly from yesterday's option price would be a biased way to test either option formula. This bias may be diminished by using a different option. Recall that the Black–Scholes model economizes on the leverage parameters by assuming they are zero.

[13]All the firm's debt could be moved to a point in time which is the average maturity for that firm's industry, or possibly this critical time could be found using duration. Morris (1975) found the average maturity for 159 industrial firms, using both short term and long term debt weighted by a percent of total value to be about 6.4 years.

Merton (1973a) proved two theorems which depended on the assumption that the return distribution of the stock is independent of the stock price level. The first theorem was that options are homogeneous of degree one in stock price and exercise price, and the second theorem showed that options are convex in the stock price. Although the stock price does not directly enter eq. (4) for the value of a call option, neither of these theorems is necessarily valid in the compound option model, since the distribution of stock returns is dependent on the level of stock prices. However, an analog to these theorems is valid if the distribution of changes in the value of the firm is independent of the level of the firm value. Following Merton (1973a) it is straightforward to show first that the value of the call is homogeneous of degree one in the value of the firm, V, the face value of the debt, M, and the call's exercise price, K, and second that options are convex in the value of the firm.[14] Since this linear homogeneity can be demonstrated without knowledge of the solution for the compound call option given in eq. (4), this property can assist in the solution by establishing its form. By Euler's theorem, since C is linearly homogeneous in V, M, and K, the solution to eq. (1) subject to eq. (2) and both boundary conditions must be of the following form:

$$C(V,t)=V\frac{\partial C}{\partial V}+M\frac{\partial C}{\partial M}+K\frac{\partial C}{\partial K}. \tag{18}$$

The solution given in eq. (4) complies with this form.

An important concept in option pricing models is the hedge ratio, defined as the partial derivative of the option price with respect to the stock price. The hedge ratio indicates the number of call contracts written (bought) against round lots of stock bought (shorted) to maintain a riskless hedge. In the compound option model this hedge ratio between the stock and the option can be found by multiplying the partial derivative of the option price with respect to the value of the firm by the reciprocal of the partial derivative of the stock price, given by eq. (3), with respect to the value of the firm, so

$$\frac{\partial C}{\partial S}=\frac{\partial C}{\partial V}\Big/\frac{\partial S}{\partial V}=\frac{N_2(h+\sigma_V\sqrt{\tau_1},\, k+\sigma_V\sqrt{\tau_2};\, \sqrt{\tau_1/\tau_2})}{N_1(k+\sigma_V\sqrt{\tau_2})}. \tag{19}$$

This differs from the hedge ratio in the Black–Scholes model,[15] which is $N_1(h+\sigma_V\sqrt{\tau_1})$, and the two are only equal when the firm has no leverage. Thus

[14]The proofs of these two theorems, given in Merton (1973a, pp. 149–150), are not repeated here. It follows that the stock is linearly homogeneous in V and M, and thus that the compound call option is also implicitly linearly homogeneous in the stock price, even though the stock's return distribution is dependent on the level of S.

[15]See Black-Scholes (1973) for a description of their hedge ratio. In h, (the integral limit of their hedge ratio), $V/\bar{V}$ is replaced by S/K and σ_V by σ_S, which is the case for an unlevered firm since $V=S$, $\bar{V}=K$, and $\sigma_V=\sigma_S$.

for options written on the equity of levered firms, if the compound option model is correct, a Black–Scholes hedge will not be riskless. It is easily verified that whenever the firm has leverage, the stock-option hedge ratio from the compound option model is greater than the Black–Scholes hedge ratio. Therefore, fewer call contracts must be written (bought) to offset long (short) round lot positions in the stock. To see this, first note that, given the value of the firm at a particular point in time, t, $N_2(\cdot)$ may be loosely interpreted as a measure of the joint probability that at t^* the value of the firm will be greater than $\bar{V}$ so that $S_{t^*}>K$ and the option is exercised, and at T the value of the firm will be greater than the face value of the debt so that the firm is not bankrupt. Since the correlation between these two events, $\sqrt{\tau_1/\tau_2}$, is always positive or zero, $N_2(\cdot)\geqq N_1(h+\sigma_V\sqrt{\tau_1})N_1(k+\sigma_V\sqrt{\tau_2})$, and the result is established. Thus if the compound option model is the correct way to capture the leverage effects and subsequent non-stationarity in the stock's instantaneous return variance, then hedgers using the Black–Scholes hedge ratio do not have riskless positions.[16] Such hedgers would be systematically over-investing in calls or under-investing in stock, depending on the hedge.

The introduction of these leverage effects adds a new dimension to theoretical option pricing. Any change in the stock price will cause a discrepancy between the compound option value and the Black–Scholes value. The qualitative discrepancies between these two formulas corresponds to what practitioners and empiricists observe in the market – namely, that the Black–Scholes formula underprices deep-out-of-the-money options and near-maturity options, and it overprices deep-in-the-money options.[17] Since options are issued near-the-money, the stock price must undergo a considerable rise or fall before either deep-in or deep-out-of-the-money options will exist. The change in the firm's leverage as the stock price changes will cause the variance of the stock in the compound option model to change in the direction necessary to alleviate these biases. This same leverage effect also acts in the proper direction to correct the time to maturity bias.

Since both the compound option and Black–Scholes models assume that the stock follows a diffusion process, the probability of deep-in and deep-out-of-the-money options existing simultaneously on one stock is small. Casual empiricism of quoted option prices substantiates this view and also offers a possible check on the frequency of stocks that may exhibit a jump process. The near-to-expiration bias complicates the problem because if the diffusion

[16]See Rosenberg (1972), Blume (1971), Black (1975), and more recently Schmalensee and Trippi (1978) for evidence on the non-stationarity of σ_s. In particular Black and Schmalensee and Trippi document the inverse relationship between S and σ_s.

[17]See Black (1975). He discusses some of the biases observed when comparing the Black–Scholes model to actual prices. Also see Black and Scholes (1972). Merton (1976) claims the model underprices deep-in-the-money options.

assumption is correct then the probable time elapsed before a stock price could diffuse along the path required for the existence of either deep-in or deep-out-of-the-money biases may also make these options 'near-to-expiration'. Furthermore, the nature of the actual conditions under which these biases are observed by market makers and empiricists is important to understanding the problem.[18] Also tests using closing prices which are possibly invalid due to market makers manipulating their margin requirements may cause these biases.

The key to whether the compound option model dominates the Black–Scholes model depends upon both model's variance assumptions. If the variance of the firm is more stationary than the variance of the firm's equity, which the compound option model predicts, then the compound option model is probably a more fundamental model.

Although the compound option model offers an explanation for these observed biases of the Black–Scholes model, it is not the only explanation. Merton (1976) showed that jump processes might explain some biases, while Geske (1978) showed that a stochastic dividend yield might explain the deep-out-of-the-money bias, and Roll (1977) showed that the dividend effect on American options might explain the deep-in-the-money bias.

4. On relationships to other option pricing results

The assumption that stock price changes follow a stationary random walk in continuous time, and thus the stock price distribution at the end of any finite interval is lognormal, is a frequent assumption in finance, especially in option pricing models. However, if the M–M theorem is correct, and if the firm is financed with risky debt, then the implication is that future stock price changes cannot be lognormal, regardless of the probability distribution for changes in the value of the firm. The reason is that with risky debt there must be some future values of V such that the firm must liquidate to pay off a portion of the debt, and for these low V's the value of the equity must be zero, which is not allowable for standard lognormal random variables. *A fortiori*, future stock price changes cannot be standard lognormal if changes in the value of the firm are assumed stationary lognormal as in the compound option model, even if the debt is riskless.

In the compound option model, the stock's return distribution depends on the stock price. Notice that since changes in the value of the firm are assumed to be stationary, it is evident from eq. (17) that the variance of stock price changes will generally be non-stationary.[19] This elasticity of the

[18] Discussions in September 1976 with market markers at the Chicago Board Options Exchange (CBOE) who use the Black–Scholes model as one guide for trading decisions revealed no concensus about how these deep-in and out-of-the-money biases are observed.

[19] Galai and Masulis (1976) reason that the firm's systematic risk, β_S is not stationary when the stock is an option on the value of the firm.

stock price with respect to the value of the firm, $\varepsilon_S=(\partial S/\partial V)(V/S)$, is always greater than or equal to one, and as V approaches zero ($S\to 0$ also), this elasticity approaches infinity. Thus, the instantaneous variance of the value of the equity is always greater than the instantaneous variance of the value of the firm.[20]

The compound option model can be related to the Cox–Ross (1975) constant elasticity of variance models.[21] In these diffusion models, the instantaneous variance of the stock price is assumed to be given by $\sigma_S^2=S^\beta\sigma^2$, where σ^2 is the instantaneous diffusion coefficient of the Weiner process and $0\leqq\beta\leqq 2$. Thus the elasticity of the variance with respect to the stock price is bounded. Since $\beta=2$ implies the diffusion process is lognormal, the Black–Scholes model is one special case of these constant elasticity of variance models. In the compound option model the elasticity of the instantaneous variance of stock returns with respect to the stock price is not a constant. However, if the elasticity of the stock price with respect to the value of the firm, ε_S, is assumed to equal a power function of S, then the compound option model reduces to a form of the constant elasticity of variance models. In particular, if $\varepsilon_S=(\partial S/\partial V)(V/S)=S^{-\gamma(V)}$, where $0<\gamma(V)<1$, then the compound option model becomes a constant elasticity of variance model,[22] where $\mathrm{d}S=\psi_S\,\mathrm{d}t+S^{\beta(V)/2}\sigma_V\,\mathrm{d}Z$ and $\beta(V)\equiv 2(1-\gamma(V))$, and the instantaneous variance of the stock price, S, is $\sigma_S^2=S^{\beta(V)}\sigma_V^2$.

Under certain situations, warrants can be treated as compound options. Since a warrant generally has a longer life than a call option, the valuation adjustments of the compound option model may be more significant for warrants than for options. Here the requirement that the expiration of the warrant be less than or equal to the maturity of the bonds may be more restrictive, particularly if an average T is used. The parameter T may cause an additional measurement problem, since it could conceivably change over time as the firm pays off old or issues new debt. Such changes in T would change the value of the firm's stock and would thus change the value of an option on the stock. If we assume that the firm matches the life of its assets with the maturity of its liabilities, replacing old with new debt so that average T remains constant over time, this remedies the problem unless shifts in technology cause changes in the average life of the firm's assets.

5. Summary

An extension to the theory for valuing contingent claims has been developed, and a new formula was derived for the value of a call option as a

[20]Repeated use of l'Hopital's rule confirms that $\lim_{V\to 0}\varepsilon_S=\infty$.

[21]For a discussion of these models, see Cox–Ross (1975) and Cox (1975).

[22]The assumption that $\varepsilon_S=S^{-\gamma}$ follows from Thorpe (1976). Since $(\partial S/\partial V)V=S^{\beta(V)/2}$ $=N_1(k+\sigma_V\sqrt{\tau_2})$, then the elasticity of the variance is $\beta(V)=2\ln(N_1(k+\sigma_V\sqrt{\tau_2}))V/\ln S$. Thorpe simulates this for various intervals of V and finds that $\beta(V)$ does vary with V and is less than 2 for the ranges tested.

compound option which introduces leverage effects into put-call option pricing. Since many corporate liabilities with sequential opportunities fit this compound option mold, their solutions can also be approached in this fashion. Thus, the theory of compound options can be used to price out the capital structure of the firm.

Appendix

Proof. In order to solve eqs. (1) and (2), subject to their boundary conditions, separate the variables and obtain equations with known Fourier integrals.[23] Thus, define $j(a,b)$ such that

$$C(V,t) \equiv \mathrm{e}^{-r_F\tau_1} j(a,b),$$

where

$$a \equiv (2/\sigma_V^2)(r_F - 1/2\sigma_V^2)(\ln(V/\bar{V}) + (r_F - 1/2\sigma_V^2)\tau_1),$$

and

$$b \equiv (2/\sigma_V^2)(r_F - 1/2\sigma_V^2)\tau_1,$$

and define $\mathrm{d}(u,p)$ such that

$$S(V,t) \equiv \mathrm{e}^{-r_F\tau_2}\mathrm{d}(u,p),$$

where

$$u \equiv (2/\sigma_V^2)(r_F - 1/2\sigma_V^2)(\ln(V/M) + (r_F - 1/2\sigma_V^2)\tau_2),$$

and

$$p \equiv (2/\sigma_V^2)(r_F - 1/2\sigma_V^2)\tau_2.$$

Substituting these definitions into eqs. (1) and (2), respectively, changes (1) into $\partial j(a,b)/\partial b = \partial^2 j(a,b)/\partial a^2$ subject to the boundary condition $j(a,0)$ equals zero if $a \leqq 0$ or $j(a,0)$ equals $K[\mathrm{e}^{a(1/2\sigma_V^2)/(r_F - 1/2\sigma_V^2)} - 1]$ if $a > 0$, and changes (2) into $\partial d(u,p)/\partial p = \partial^2 d(u,p)/\partial u^2$ subject to the boundary condition $d(u,0) = 0$ if $u \leqq 0$ or $d(u,0)$ equals $M[\mathrm{e}^{u(1/2\sigma_V^2)/(r_F - 1/2\sigma_V^2)} - 1]$ if $u > 0$. Implementing the Fourier integrals, rechanging the variables, and substituting the solution given in eq. (3) for $S(V,t)$ into $C(V,t)$ yields the following equation:

[23]See Churchill (1963, pp. 152–156) for reference.

$$C(V,t)=V\int_{-a/\sqrt{2b}}^{\infty}\int_{-u/\sqrt{2p}}^{\infty}\frac{e^{-\frac{1}{2}(q-\sigma_V\sqrt{\tau_2})^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}(z-\sigma_V\sqrt{\tau_1})^2}}{\sqrt{2\pi}}\,dq\,dz$$
$$-M e^{-r_F\tau_2}\int_{-a/\sqrt{2b}}^{\infty}\int_{b/\sqrt{2p}}^{\infty}\frac{e^{-\frac{1}{2}q^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}(z-\sigma_V\sqrt{\tau_1})^2}}{\sqrt{2\pi}}\,dq\,dz$$
$$-K e^{-r_F\tau_1}\int_{-a/\sqrt{2b}}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz.$$

To evaluate these three integrals, note that the bivariate normal distribution function is the definite integral solution to the following density function, where x and y are any two bivariate normal random variables and ρ is their correlation:

$$N_2(h,k;\rho)=\int_{-\infty}^{h}\int_{-\infty}^{k}f(x,y)\,dx\,dy=\int_{-\infty}^{h}\int_{-\infty}^{k}f(y|x)f(x)\,dy\,dx$$
$$=\frac{1}{2\pi\sqrt{1-\rho^2}}\int^{h}\int_{-\infty}^{k}\exp\left\{-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)\right\}dx\,dy.$$

Defining $y=w\sqrt{1-\rho^2}+\rho x$ and changing variables yields

$$N_2(h,k;\rho)=\int_{-\infty}^{h}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\left[\int_{-\infty}^{k-\rho x/\sqrt{1-\rho^2}}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}\,dw\right]dx$$
$$=\int_{-\infty}^{h}f(x)N_1\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)dx.$$

Thus, evaluating the above three integrals[24] yields eq. (4). Q.E.D.

[24]For reference see Abramowitz and Stegum (1970) or Owen (1957).

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